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Solution of ordinary differential equations via nonlocal transformations

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Abstract
The classical Lie analysis provides a useful technique for the solution of ordinary differential equations via point symmetries/transformations. Unfortunately, the requirement that an nth-order equation possesses at least an n-dimensional solvable Lie algebra of symmetries is only satisfied by a select number of equations. We provide a class of second-order ordinary differential equations with less than the required two Lie point symmetries that can be solved via nonlocal transformations.

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1. Introduction

The Lie theory of extended groups applied to ordinary differential equations is one of the most successful techniques for the solution of these equations. Once the appropriate number of symmetries have been determined for a particular equation, the route to its solution follows a well defined algorithm [17]. However, few equations admit the required number of point symmetries to enable reduction to quadratures.

In an attempt to overcome this limitation, various extensions of the classical Lie approach have been devised. One such extension is due to the observance of *hidden* symmetries (i.e. point symmetries that arise unexpectedly in the increase/decrease of order of an equation) by Olver [17] which was systematically investigated (and named) by Abraham-Shrauner and Guo [2]. These symmetries have their origin in nonlocal symmetries of the original equation. (The idea of extending the local applicability of Lie groups is not new. An early consideration is due to Mostow [15].)

Clearly, the identification of nonlocal symmetries of equations would suggest that they can be reduced to quadratures despite the lack of point symmetries. However, it is difficult to

Table 1. 3D Lie algebras appropriate to third-order equations.

Type	Nonzero commutation relations		
$3A_1$			
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$		
$A_{3,1}$	$[G_2, G_3] = G_1$		
$A_{3,2}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = G_1 + G_2$	
$A_{3,3}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = G_2$	
$A_{3,4}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = -G_2$	
$A^a_{3,5}$ $(0 < \vert a\vert < 1)$	$[G_1, G_3] = G_1$	$[G_2, G_3] = aG_2$	
$A_{3,6}$	$[G_1, G_3] = -G_2$	$[G_2, G_3] = G_1$	
$A^b_{3,7}$ $(b > 0)$	$[G_1, G_3] = bG_1 - G_2$	$[G_2, G_3] = G_1 + bG_2$	
$A_{3,8}$	$[G_1, G_2] = G_1$	$[G_2, G_3] = G_3$	$[G_3, G_1] = -2G_2$
$A_{3,9}$	$[G_1, G_2] = G_3$	$[G_2, G_3] = G_1$	$[G_3, G_1] = G_2$

directly calculate nonlocal symmetries although some ideas were presented in [8]. The most successful techniques have involved indirect methods [3, 9].

The idea behind [9] arose as a result of the solution of a second-order equation not possessing Lie point symmetries [1]. This second-order equation was shown to be linked (via a nonlocal transformation) to another second-order equation possessing two point symmetries. As a result, the original equation could be solved. Govinder and Leach [9] classified second-order equations not possessing Lie point symmetries using this approach. Adam and Mahomed [3] proceeded in a similar manner, but confined their work to first-order equations.

Here we hope to fill the gap between those two bodies of work. We wish to relate second-order equations with fewer than two point symmetries to those with at least two point symmetries. This will enable the original equations to be solved. Our approach will be via third-order equations. We consider all third-order equations invariant under three-dimensional Lie algebras. Each equation will be reduced via its point symmetries and the symmetries of the resulting second-order equations will be investigated. The nonlocal transformations between those reduced equations with fewer than two point symmetries and those with at least two point symmetries will then be determined.

2. Nonlocal transformations of second-order ODEs

We undertake a systematic reduction of all third-order equations invariant under three-dimensional Lie algebras. In what follows we utilize the Mubarakzyanov classification [16] scheme as explained in [13, 14, 18]. We take our equations and Lie algebras from the latter works. They have recently appeared in [11]. (Note that the forms of some of the algebras listed below do not conform exactly to those in [18]: e.g., $A_1 \oplus A_2$ and $A_{3,8}$. However, a simple change of basis will bring both tables into agreement.)

There are 11 real three-dimensional Lie algebras (see table 1). We ignore $3A_1$ and one representation of $A_{3,3}$ (i.e. $A^{II}_{3,3}$) as the equations invariant under these Lie algebras are linear. As a result those equations admit larger classes of Lie algebras. The equation invariant under $A^I_{3,8}$ admits a six-dimensional Lie algebra and also falls outside the scope of this paper. (This observation is omitted in [11, 12] where the equation is presented as being invariant under $A^I_{3,8}$ only.) The same is true for the equation invariant under $A^I_{3,9}$ while $A^{II}_{3,9}$ does not admit a third-order equation.

We illustrate the general procedure via $A_{3,1}$: this algebra admits the equation

$$y''' = \Gamma[y''] \tag{1}$$

and has the following canonical symmetry representation:

$$G_1 = \frac{\partial}{\partial y} \tag{2}$$

$$G_2 = \frac{\partial}{\partial x} \tag{3}$$

$$G_3 = x\frac{\partial}{\partial y}. \tag{4}$$

Using each of these three symmetries, we reduce equation (1) to second-order equations.

Reduction via G_1 yields

$$u = x \qquad v = y' \tag{5}$$

$$y'' = v' \tag{6}$$

$$y''' = v''. \tag{7}$$

Equation (1) becomes

$$v'' = \Gamma[v']. \tag{8}$$

The reduction variables obtained via G_1 result in the symmetries G_2 and G_3 transforming to

$$X_2 = \frac{\partial}{\partial u} \tag{9}$$

$$X_3 = \frac{\partial}{\partial v}. \tag{10}$$

Since both are point symmetries of (8) we can reduce this equation to quadratures.

Reduction via G_2 yields

$$u = y \qquad v = y' \tag{11}$$

$$y'' = vv' \tag{12}$$

$$y''' = v^2v'' + v(v')^2. \tag{13}$$

Thus (1) reduces to

$$v'' = -\frac{(v')^2}{v} + \frac{1}{v^2}\Gamma[vv']. \tag{14}$$

From G_1 we obtain

$$X_1 = \frac{\partial}{\partial u} \tag{15}$$

but G_3 cannot be rewritten in terms of the new coordinates as a local symmetry. As a result, (14) cannot be directly reduced to quadratures.

Finally considering G_3 we obtain

$$u = x \qquad v = y' - \frac{y}{x} \tag{16}$$

$$y'' = v' + \frac{v}{u} \tag{17}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2}. \tag{18}$$

The reduced equation is

$$v'' = \frac{v}{u^2} - \frac{v'}{u} + \Gamma\left[v' + \frac{v}{u}\right]. \tag{19}$$

Table 2. Second-order equations admitting 2D Lie algebras.

Type	$[G_1, G_2]$	Symmetries	Invariant equation
I	0	$G_1 = \frac{\partial}{\partial T}$	$Q'' = F(Q')$
		$G_2 = \frac{\partial}{\partial Q}$	
II	0	$G_1 = \frac{\partial}{\partial Q}$	$Q'' = F(T)$
		$G_2 = T\frac{\partial}{\partial Q}$	
III	G_1	$G_1 = \frac{\partial}{\partial Q}$	$TQ'' = F(Q')$
		$G_2 = T\frac{\partial}{\partial T} + Q\frac{\partial}{\partial Q}$	
IV	G_1	$G_1 = \frac{\partial}{\partial Q}$	$Q'' = Q'F(T)$
		$G_2 = Q\frac{\partial}{\partial Q}$	

G_1 takes on the new form

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v}. \tag{20}$$

Once again, reduction results in the *loss* of a point symmetry, this time G_2. This implies that (19) cannot be directly reduced to quadratures.

We have obtained one second-order equation that can be directly reduced to quadratures and two which cannot from a single third-order equation. To enable us to reduce the latter two equations to quadratures, we must first effect nonlocal transformations into the former equation. These transformations are given by

$$\begin{aligned} u_2 &= \int v_1 \, \mathrm{d}u_1 \\ v_2 &= v_1 \end{aligned} \tag{21}$$

and

$$\begin{aligned} u_3 &= u_1 \\ v_3 &= v_1 - \frac{\int v_1 \, \mathrm{d}u_1}{u_1}. \end{aligned} \tag{22}$$

(Here the variables u_i and v_i refer to the reduction variables obtained from the symmetry G_i.)

To fully utilize the above procedure, we need to investigate the solution of the second-order equations admitting two-dimensional Lie algebras. There are two 2D Lie algebras (each admitting two representations) (see table 2). The solutions of these equations can be easily calculated [10]. We recall those for easy reference.

In the case of type I equations, the solution is given by

$$Q = \int \phi(T + c_0) \, \mathrm{d}T + c_1 \tag{23}$$

where the c_i are constants of integration and ϕ is obtained by solving

$$\int \frac{\mathrm{d}Q'}{F(Q')} = T + c_0 \tag{24}$$

for Q'. For type II equations the solution is

$$Q = \int \left(\int F(T) \, \mathrm{d}T \right) \mathrm{d}T + c_1 T + c_0. \tag{25}$$

In the case of type III equations the solution is

$$Q = \int \phi(\ln T + c_0)\,\mathrm{d}T + c_1 \tag{26}$$

where ϕ is obtained by solving

$$\int \frac{\mathrm{d}Q'}{F(Q')} = \ln T + c_1 \tag{27}$$

for Q' and for type IV we have

$$Q = c_1 \int \exp\left(\int F(T)\,\mathrm{d}T\right)\mathrm{d}T + c_0. \tag{28}$$

In the case of (8), we replace u with T and v with Q and utilize (23) as solution.

We apply the above procedure to all appropriate three-dimensional Lie algebras. Below follows a list of the second-order equations with fewer than two symmetries that can be solved via nonlocal transformations. Each set is followed by the nonlocal transformation linking them to an equation(s) with two point symmetries. The transformation between one of the equations with two symmetries and the forms in table 2 is provided so that the route to solution is clear. In addition, the form of F (needed in table 2) is given in terms of the new variables and Γ.

Algebra $A_1 \oplus A_2^I$

$$v_3'' = -\frac{(v_3')^2}{v_3} + \frac{3v_3'}{v_3} - \frac{2}{v_3} + \frac{(v_3'-1)^{\frac{3}{2}}}{v_3^{\frac{1}{2}}}\Gamma\left[\frac{v_3'-1}{v_3}\right] \tag{29}$$

$$u_3 = u_1 \qquad v_3 = v_1 \int \frac{1}{v_1}\,\mathrm{d}u_1 \tag{30}$$

$$u_3 = \int v_2\,\mathrm{d}u_2 \qquad v_3 = u_2 v_2 \tag{31}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \left(\frac{(v_1')^3}{v_1}\right)^{\frac{1}{2}}\Gamma\left[\frac{v_1'}{v_1}\right] \tag{32}$$

$$v_2'' = v_2'^{\frac{3}{2}}\Gamma\left[\frac{v_2'}{v_2^2}\right]. \tag{33}$$

The latter equation is of type III. We employ the transformation

$$Q = u_2 \qquad T = \frac{1}{v_2} \tag{34}$$

with

$$F(Q') = -2Q' + Q'\Gamma\left[-\frac{1}{Q'}\right] \tag{35}$$

and the solution is given by (26).

Algebra $A_1 \oplus A_2^{II}$

$$v_3'' = \frac{-(v_3')^2 + 2v_3'}{v_3 - u_3} + (v_3')^2\Gamma[(v_3 - u_3)v_3'] \tag{36}$$

$$u_3 = \frac{\int v_1\,\mathrm{d}u_1}{u_1} \qquad v_3 = v_1 \tag{37}$$

$$u_3 = \int \frac{v_2}{u_2}\,\mathrm{d}u_2 \qquad v_3 = v_2 + \int \frac{v_2}{u_2}\,\mathrm{d}u_2 \tag{38}$$

$$v_1'' = (v_1')^2\Gamma[u_1 v_1'] \tag{39}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' + \frac{v_2}{u_2}\right)^2 \Gamma[u_2 v_2' + v_2]. \tag{40}$$

This time, the former equation is of type I. The transformation used is

$$T = v_1 \qquad Q = \log u_1 \tag{41}$$

with

$$F(Q') = -Q'^2 - Q'\Gamma\left[\frac{1}{Q'}\right]. \tag{42}$$

The solution is given by (23).

Algebra $A_{3,1}$

$$v_2'' = -\frac{(v_2')^2}{v_2} + \frac{1}{v_2^2}\Gamma[v_2 v_2'] \tag{43}$$

$$v_3'' = \frac{v_3}{u_3^2} - \frac{v_3'}{u_3} + \Gamma\left[v_3' + \frac{v_3}{u_3}\right] \tag{44}$$

$$u_2 = \int v_1\,\mathrm{d}u_1 \qquad v_2 = v_1 \tag{45}$$

$$u_3 = u_1 \qquad v_3 = v_1 - \frac{\int v_1\,\mathrm{d}u_1}{u_1} \tag{46}$$

$$v_1'' = \Gamma[v_1']. \tag{47}$$

This equation is of type I. Setting

$$T = u_1 \qquad Q = v_1 \tag{48}$$

with

$$F(Q') = \Gamma[Q'] \tag{49}$$

allows one to obtain the solution from (23).

Algebra $A_{3,2}^I$

$$v_3'' = \frac{-(v_3')^2 + 2v_3'}{v_3 - u_3 - 1} + \frac{1}{(v_3 - u_3 - 1)^2} + \left[\frac{[(v_3 - u_3 - 1)v_3' + 1]^2}{(v_3 - u_3 - 1)^2}\right]\Gamma[\exp(v_3(v_3 - u_3 - 1)v_3' + 1)] \tag{50}$$

$$u_3 = \frac{\int v_1\,\mathrm{d}u_1}{u_1} - \log u_1 \qquad v_3 = v_1 - \log u_1 \tag{51}$$

$$u_3 = \frac{u_2}{\int \frac{1}{v_2}\,\mathrm{d}u_2} - \log\left[\int \frac{1}{v_2}\,\mathrm{d}u_2\right] \qquad v_3 = v_2 - \log\left[\int \frac{1}{v_2}\,\mathrm{d}u_2\right] \tag{52}$$

$$v_1'' = (v_1')^2\Gamma[v_1' \exp v_1] \tag{53}$$

$$v_2'' = -\frac{(v_2')^2}{v_2} + (v_2')^2\Gamma[v_2 v_2' \exp v_2]. \tag{54}$$

The former equation is of type III. The transformation we employ is

$$Q = u_1 \qquad T = \mathrm{e}^{v_1} \tag{55}$$

with F given by

$$F(Q') = -Q'\left(1 + \Gamma\left[\frac{1}{Q'}\right]\right) \tag{56}$$

and the solution by (26).

Algebra $A_{3,2}^{II}$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' + \frac{v_2}{u_2}\right)\Gamma\left[\frac{\exp u_2}{v_2' + \frac{v_2}{u_2}}\right] \tag{57}$$

$$v_3'' = \frac{(v_3')^2}{1 - v_3} + \frac{v_3}{(v_3 - 1)}\left[\frac{v_3^2}{(v_3 - 1)} + v_3'\right] + \frac{[(v_3 - 1)v_3' + v_3^2]}{(v_3 - 1)^2}\Gamma\left[\frac{\exp(-u_3)}{(v_3 - 1)v_3' + v_3^2}\right] \tag{58}$$

$$u_2 = u_1 \qquad v_2 = v_1 - \frac{\int v_1\,\mathrm{d}u_1}{u_1} \tag{59}$$

$$u_3 = \log\left[\int v_1\,\mathrm{d}u_1\right] - u_1 \qquad v_3 = \frac{v_1}{\int v_1\,\mathrm{d}u_1} \tag{60}$$

$$v_1'' = v_1'\Gamma\left[\frac{\exp u_1}{v_1'}\right]. \tag{61}$$

This equation is of type III. We utilize

$$Q = v_1 \qquad T = \mathrm{e}^{u_1} \tag{62}$$

with

$$F(Q') = -Q'\left(1 - \Gamma\left[\frac{1}{Q'}\right]\right) \tag{63}$$

and the solution is given by (26).

Algebra $A_{3,3}^{I}$

$$v_3'' = \frac{-(v_3')^2 + 2v_3'}{(v_3 - u_3)} + (v_3')^2\Gamma[v_3] \tag{64}$$

$$u_3 = \frac{u_1}{\int \frac{1}{v_1}\,\mathrm{d}u_1} \qquad v_3 = v_1 \tag{65}$$

$$u_3 = \frac{\int v_2\,\mathrm{d}u_2}{u_2} \qquad v_3 = v_2 \tag{66}$$

$$v_1'' = \frac{-(v_1')^2}{v_1} + (v_1')^2\Gamma[v_1] \tag{67}$$

$$v_2'' = (v_2')^2\Gamma[v_2]. \tag{68}$$

The latter equation is of type IV and is hence linearizable. All we need do is interchange the independent and dependent variables, namely

$$Q = u_2 \qquad T = v_2 \tag{69}$$

with

$$F(T) = -\Gamma[T]. \tag{70}$$

The solution is given by (28).

Algebra $A_{3,4}^{I}$

$$v_3'' = -\frac{u_3(v_3')^2 + 2v_3v_3' - 6}{u_3v_3+1} + \frac{18v_3^3}{(u_3v_3+1)^2} + \frac{((u_3v_3+1)v_3' + 2v_3^2)^{\frac{4}{3}}}{(u_3v_3+1)^2} \times\Gamma\left[\left(\frac{u_3}{v_3^{\frac{1}{2}}} + \frac{1}{v_3^{\frac{3}{2}}}\right)v_3' + 2v_3^{\frac{1}{2}}\right] \tag{71}$$

$$u_3 = u_1\int\frac{1}{v_1}\,\mathrm{d}u_1 \qquad v_3 = \frac{v_1}{u_1^2} \tag{72}$$

$$u_3 = u_2\int v_2\,\mathrm{d}u_2 \qquad v_3 = \frac{v_2}{(\int v_2\,\mathrm{d}u_2)^2} \tag{73}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \left(\frac{(v_1')^2}{v_1}\right)^{\frac{2}{3}}\Gamma\left[\frac{v_1'}{v_1^{\frac{1}{2}}}\right] \tag{74}$$

$$v_2'' = (v_2')^{\frac{4}{3}}\Gamma\left[\frac{v_2'}{v_2^{\frac{3}{2}}}\right]. \tag{75}$$

The former equation is of type III. Here we use

$$Q = u_1 \qquad T = (v_1)^{1/2} \tag{76}$$

with

$$F(Q') = Q' - \frac{1}{2}(Q')^{\frac{5}{3}}\Gamma\left[\frac{2}{Q'}\right] \tag{77}$$

and the solution is given by (26).

Algebra $A_{3,4}^{II}$

$$v_3'' = \frac{3v_3v_3'}{u_3(2v_3-u_3)} - \frac{3v_3^3}{u_3^2(2v_3-u_3)^2} - \frac{(u_3-v_3)^{\frac{10}{3}}}{u_3^2(2v_3-u_3)}\Gamma\left[\frac{(2v_3-u_3)v_3'}{u_3^{\frac{1}{2}}} - \frac{v_3^2}{u_3^{\frac{3}{2}}}\right] \tag{78}$$

$$u_3 = \frac{(\int v_1\,\mathrm{d}u_1)^2}{u_1} \qquad v_3 = v_1\int v_1\,\mathrm{d}u_1 \tag{79}$$

$$u_3 = \frac{(u_2\int\frac{v_2}{u_2}\,\mathrm{d}u_2)^2}{u_2} \qquad v_3 = \left(v_2 + \int\frac{v_2}{u_2}\,\mathrm{d}u_2\right)\left(u_2\int\frac{v_2}{u_2}\,\mathrm{d}u_2\right) \tag{80}$$

$$v_1'' = v_1'^{\frac{5}{3}}\Gamma[u_1^{\frac{3}{2}}v_1'] \tag{81}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' + \frac{v_2}{u_2}\right)^{\frac{5}{3}}\Gamma[u_2^{\frac{3}{2}}v_2' + u_2^{\frac{1}{2}}v_2]. \tag{82}$$

Here, the latter equation is of type III. We utilize

$$T = (u_2)^{1/2} \qquad Q = u_2v_2 \tag{83}$$

with

$$F(Q') = 3Q' + (2Q'^5)^{(1/3)}\Gamma\left[\frac{Q'}{2}\right] \tag{84}$$

and the solution is given by (26).

Algebra $A_{3,5}^{aI}$

$$v_3'' = -\frac{(v_3')^2 + (a-3)v_3'}{v_3 - au_3} - \frac{(a-1)(a-2)v_3}{(v_3-au_3)^2} + (v_3 - au_3)^{\frac{1-a}{a-2}}\left[v_3' + \frac{(a-1)v_3}{v_3-au_3}\right]^{\frac{a-3}{a-2}} \times\Gamma\left[v^{\frac{1}{a-1}}\left(\left(1-\frac{au_3}{v_3}\right)v_3' + a - 1\right)\right] \tag{85}$$

$$u_3 = \frac{u_1}{(\int \frac{1}{v_1}\,\mathrm{d}u_1)^a} \qquad v_3 = \frac{v_1}{(\int \frac{1}{v_1}\,\mathrm{d}u_1)^{a-1}} \tag{86}$$

$$u_3 = \frac{\int v_2\,\mathrm{d}u_2}{u_2^a} \qquad v_3 = \frac{v_2}{u_2^{a-1}} \tag{87}$$

$$v_1'' = -\frac{v_1'}{v_1} + \frac{(v_1v_1')^{\frac{a-3}{a-2}}}{v_1^2}\Gamma[v_1'v_1^{\frac{1}{a-1}}] \tag{88}$$

$$v_2'' = v_2'^{\frac{a-3}{a-2}}\Gamma[v_2'v_2^{\frac{2-a}{a-1}}]. \tag{89}$$

The latter is of type III. We utilize

$$T = (v_2)^{1/(a-1)} \qquad Q = u_2 \tag{90}$$

with

$$F(Q') = (a-2)Q' - \frac{(Q')^{\frac{2a-3}{a-2}}}{(a-1)^{\frac{1}{a-2}}}\Gamma\left[\left(\frac{(Q')^{2a-3}}{a-1}\right)^{\frac{1}{a-2}}\right] \tag{91}$$

and the solution is given by (26).

Algebra $A_{3,5}^{\frac{1}{2}I}$

$$v_3'' = -(v_3' - 1)v_3' + \frac{1}{v_3(v_3-2u_3)}\Gamma[(v_3 - 2u_3)v_3' + v_3] \tag{92}$$

$$u_3 = \frac{u_1}{(\int \frac{1}{v_1}\,\mathrm{d}u_1)^2} \qquad v_3 = \frac{v_1}{\int \frac{1}{v_1}\,\mathrm{d}u_1} \tag{93}$$

$$u_3 = \frac{\int v_2\,\mathrm{d}u_2}{u_2^2} \qquad v_3 = \frac{v_2}{u_2} \tag{94}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \frac{1}{v_1^3}\Gamma[v_1v_1'] \tag{95}$$

$$v_2'' = \frac{1}{v_2}\Gamma[v_2']. \tag{96}$$

Once again the latter equation is of type III. Here we choose

$$T = \frac{1}{v_2} \qquad Q = u_2 \tag{97}$$

and set

$$F(Q') = -Q'^3\Gamma\left[\frac{1}{Q'}\right]. \tag{98}$$

The solution is given by (26).

Algebra $A_{3,5}^{aII}$

$$\begin{aligned}v_3'' = &-\frac{(1-a)(v_3')^2}{[(1-a)v_3 - u_3]} - \frac{[(1-a)(3av_3^2 + 2u_3^2(u_3 - v_3)) - au_3^2 v_3]v_3'}{u_3^2[(1-a)v_3 - u_3]^2} \\ &-\frac{av_3^3(2a-1)}{u_3^2[(1-a)v_3 - u_3]^2} + \frac{[(1-a)u_3 v_3 v_3' + u_3^2 v_3' + av_3^2]^{\frac{2-3a}{1-2a}}}{u_3^2[(1-a)v_3 - u_3]^2} \\ &\times\Gamma[v_3 v_3'(1-a)u_3^{\frac{a}{1-a}} - u_3^{\frac{1}{1-a}} v_3' + av_3^2 u_3^{\frac{2a-1}{1-a}}]\end{aligned} \tag{99}$$

$$u_3 = \frac{[\int v_1\,\mathrm{d}u_1]^{1-a}}{u_1} \qquad v_3 = \frac{v_1}{[\int v_1\,\mathrm{d}u_1]^a} \tag{100}$$

$$u_3 = \frac{[u_2\int \frac{v_2}{u_2}\,\mathrm{d}u_2]^{1-a}}{u_2} \qquad v_3 = \frac{v_2 + \int \frac{v_2}{u_2}\,\mathrm{d}u_2}{[u_2\int \frac{v_2}{u_2}\,\mathrm{d}u_2]^a} \tag{101}$$

$$v_1'' = v_1'^{\frac{2-3a}{1-2a}}\Gamma[u_1^{\frac{2a-1}{a-1}} v_1'] \tag{102}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' - \frac{v_2}{u_2}\right)^{\frac{2-3a}{1-2a}}\Gamma[u_2^{\frac{2a-1}{a-1}} v_2' + v_2 u_2^{\frac{a}{a-1}}]. \tag{103}$$

The former equation is of type III. We employ the transformation

$$T = (u_1)^{a/(1-a)} \qquad Q = v_1 \tag{104}$$

with

$$F(Q') = \frac{1-2a}{a}Q' + \left(\frac{1-a}{a}\right)^{a/(2a-1)}(Q')^{\frac{2-3a}{1-2a}}\Gamma\left[\frac{a}{1-a}Q'\right] \tag{105}$$

and the solution is given by (26).

Algebra $A_{3,5}^{\frac{1}{2}II}$

$$v_3'' = \frac{(-(v_3')^2 + v_3')}{(v_3 - 2u_3)} + \frac{1}{(v_3 - 2u_3)^2}\Gamma[(v_3 - 2u_3)v_3' + v_3] \tag{106}$$

$$u_3 = \frac{\int v_1\,\mathrm{d}u_1}{u_1^2} \qquad v_3 = \frac{v_1}{u_1} \tag{107}$$

$$u_3 = \frac{\int \frac{v_2}{u_2}\,\mathrm{d}u_2}{u_2} \qquad v_3 = \frac{v_2 + \int \frac{v_2}{u_2}\,\mathrm{d}u_2}{u_2} \tag{108}$$

$$v_1'' = \frac{1}{u_1}\Gamma[v_1'] \tag{109}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \frac{1}{u_2}\Gamma\left[v_2' + \frac{v_2}{u_2}\right]. \tag{110}$$

The latter equation is of type III. We use the transformation

$$T = u_2^2 \qquad Q = u_2 v_2 \tag{111}$$

with

$$F(Q') = \frac{\Gamma[2Q']}{4}. \tag{112}$$

The solution is given by (26).

3. Discussion

We submit this list of equations as a contribution to the class of second-order ordinary differential equations that can be reduced to quadratures. It remains to consider other third-order equations invariant under larger (>3)-dimensional Lie groups. A list of the relevant equations appears in [11]. This work is ongoing.

Here we have been interested in producing second-order equations that can be reduced to quadratures in spite of the sparsity of point symmetries. We have not focused on the actual solution of the third-order equations as they have merely been utilized as a tool in our calculations. The study of third-order equations [5, 12] and their first integrals [6, 7] has also received attention.

While point symmetries of differential equations have always played (and we believe will continue to play) an important role in the reduction of order of equations, it is evident that nonlocal transformations (symmetries) also have their special role to play. The results in [3, 9] and those of this paper testify to this.

As a final remark we note that most results dealing with nonlocal transformations or symmetries have been obtained in a purely theoretic setting. However, recently [4] nonlocal transformations have been utilized in the solution of the Einstein field equations. Hopefully, the classes of solutions presented here will have similar applications.

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